Phys 410
Spring 2013
Lecture #18 Summary
4 March, 2013

The calculus of variations is used to find extremum values of integral functionals. An example is a calculation of the shortest distance between two points in a plane. One can write the distance in terms of an integral over the path from the designated starting point $(x_1 y_1)$ to the designated end point $(x_2 y_2)$ as $L = \int_1^2 ds = \int_1^2 \sqrt{dx^2 + dy^2}$. If we treat the x coordinate as the independent variable we can write the integral as $L = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$, where we have written $(dy/dx)^2$ as $(y')^2$. Our objective is to find the path $y(x)$ that minimizes this integral. This is a problem in the calculus of variations.

A second example is Fermat's principle. This is the problem of how light propagates from point 1 to point 2 through a variable dielectric medium characterized by an index of refraction that varies with position in a plane as $n(x, y)$. The light moves with variable speed $v = c/n(x, y)$. Fermat's principle says that light will take the path that minimizes the time to travel between the two points: $time(1 \to 2) = \frac{1}{c} \int_{x_1}^{x_2} n(x, y)\sqrt{1 + (y')^2}\, dx$. Again we need to find the path $y(x)$ that minimizes this integral. This is another problem in the calculus of variations.

The Euler-Lagrange equation is derived by assuming that there is an infinite family of "wrong" trajectories between points 1 and 2 parameterized by the function $\eta(x)$ and the constant α as $Y(x) = y(x) + \alpha\eta(x)$. The objective is to minimize the integral $S = \int_{x_1}^{x_2} f[y(x), y'(x), x]\, dx$, and this will be accomplished by taking $dS/d\alpha$ and setting it equal to zero. The result, after integrating by parts, is that the following expression must be satisfied for all points $x_1 \le x \le x_2$: $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$, called the Euler-Lagrange equation.

Going back to the shortest-distance-in-a-plane problem, we see that the function f in this case is $f = \sqrt{1 + (y')^2}$. In this case f does not depend explicitly on y, hence we can write $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}} = C$, a constant. This can be reduced to $y'(x) = m$, another constant. Integrating both sides with respect to x, we find $y(x) = mx + b$, which is the famous equation for a straight line. The Fermat's principle problem can be solved in a similar way once the index of refraction distribution $n(x, y)$, and the end points, are specified.